Practices before the class (March 3)

We will use a few minutes before the class to practice some questions, especially True/False questions. Answers will be provided during/after the class, depending on how much time we have for the lecture.

- (**T/F**) If A is $m \times n$ and rank $A = m$, then the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
- (**T/F**) If A is $m \times n$ and the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto, then $\operatorname{rank} A = m$.
- (**T/F**) If H is a subspace of $\mathbb{R}^3$, then there is a 3×3 matrix A such that $H = \operatorname{Col} A$.
- (**T/F**) If B is obtained from a matrix A by several elementary row operations, then $\operatorname{rank} B = \operatorname{rank} A$.

Practices before the class with answers (March 3)

If $A\vec{x}=\vec{b}$ has a solution, then it is unique.

- (**T/F**) If A is $m \times n$ and rank $A = m$, then the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
  False. Counterexample: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. If rank $A = n$(the number of columns in A), then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

  $A\vec{x}=\vec{b}$ has solution(s) for all $\vec{b}$

- (**T/F**) If A is $m \times n$ and the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto, then rank $A = m$.
  True. If $\mathbf{x} \mapsto A\mathbf{x}$ is onto, then Col $A = \mathbb{R}^m$ and rank $A = m$. See Theorem 12 (a) in Section 1.9.
- (**T/F**) If H is a subspace of $\mathbb{R}^3$, then there is a 3×3 matrix A such that $H = \text{Col}\, A$.
  True. If H is the zero subspace, let A be the 3×3 zero matrix.
  If dim $H = 1$, let $\{\mathbf{v}\}$ be a basis for H and set $A = \begin{bmatrix} \mathbf{v} & \mathbf{v} & \mathbf{v} \end{bmatrix}$.
  If dim $H = 2$, let $\{\mathbf{u}, \mathbf{v}\}$ be a basis for H and set $A = \begin{bmatrix} \mathbf{u} & \vec{u} & \mathbf{v} \end{bmatrix}$, for example.
  If dim $H = 3$, then $H = \mathbb{R}^3$, so A can be any 3×3 invertible matrix. Or, let $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ be a basis for H and set $A = \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix}$.
- (**T/F**) If B is obtained from a matrix A by several elementary row operations, then rank $B = \text{rank}\, A$.
  True. Row equivalent matrices have the same number of pivot columns.

5.1 Eigenvectors and Eigenvalues

Example 0. Let $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. The images of $\mathbf{u}$ and $\mathbf{v}$ under multiplication by A are shown in the following figure. In fact, $A\mathbf{v}$ is just $2\mathbf{v}$. So A only "stretches" or dilates $\mathbf{v}$.

$A\vec{u} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$

$A\vec{v} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2\vec{v}$

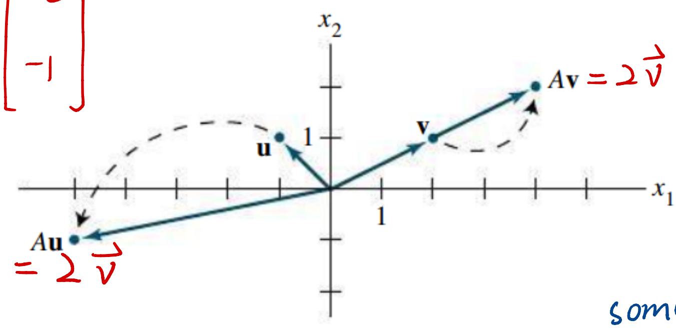


FIGURE 1 Effects of multiplication by A.

some other scalar

In this section, we study the equation like $A\vec{v} = 2\vec{v}$ i.e. the vectors are transformed by A into a scalar of themselves.

Definition.

An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an **eigenvector** corresponding to λ.

Example 1. (1) Is $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \vec{x}$ an eigenvector of $\begin{bmatrix} 2 & 6 & 7 \\ 3 & 2 & 7 \\ 5 & 6 & 4 \end{bmatrix} = A$? If so, find the eigenvalue.

By definition, the question is asking is $A\vec{x}$ a scalar multiple of $\vec{x}$?

Compute $A\vec{x} = \underbrace{\begin{bmatrix} 2 & 6 & 7 \\ 3 & 2 & 7 \\ 5 & 6 & 4 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}}_{\vec{x}} = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix} = (-3)\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = -3\vec{x}$

So $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is an engenvector of A for the eigenvalue -3.

(2) Is $\lambda = 3$ an eigenvalue of $\begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$? If so, find one corresponding eigenvector.

$$A\vec{x} = \lambda\vec{x} \iff A\vec{x} - \lambda\vec{x} = \vec{0} \iff A\vec{x} - \lambda I\vec{x} = \vec{0} \iff (A - \lambda I)\vec{x} = \vec{0}$$

To determine if 3 is an eigenvalue of A, we need to show the equation $(A-3I)\vec{x} = \vec{0}$ has nontrivial solution.

The coefficient matrix is

$$A - 3I = \begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 2 & 2 \\ 3 & -5 & 1 \\ 0 & 1 & -2 \end{bmatrix}.$$

The augmented matrix is

$$[A - 3I \quad \vec{0}] = \begin{bmatrix} -2 & 2 & 2 & 0 \\ 3 & -5 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So x_1 x_2 are basic variables and x_3 is free.

Thus if $\lambda = 3$, $(A-3I)\vec{x} = \vec{0}$ has nontrivial solution.

Therefore $\lambda = 3$ is an eigenvalue.

$$\begin{cases} x_1 = 3x_3 \\ x_2 = 2x_3 \\ x_3 \text{ is free} \end{cases} \qquad \vec{x} = x_3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

We can take $\vec{v} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ as an eigenvector corresponds to the eigenvalue $\lambda = 3$.

Remark (Eigenspaces).

- Given a particular eigenvalue λ of the n by n matrix A, define the set E to be all vectors $\mathbf{v}$ that satisfy Equation $(A-\lambda I)\mathbf{v}=\mathbf{0}$, i.e., $E=\{\mathbf{v}:(A-\lambda I)\mathbf{v}=\mathbf{0}\}$.
- Note that E equals the nullspace of the matrix $A-\lambda I$.
- E is called the eigenspace of A associated with λ.

Example 2. Find a basis for the eigenspace corresponding to each listed eigenvalue.

$$A=\begin{bmatrix}3 & -1 & 3\\ -1 & 3 & 3\\ 6 & 6 & 2\end{bmatrix}, \lambda=-4$$

ANS: The question is asking us to find a basis for the nullspace of the matrix $A-(-4)I$ from the above discussion.

We compute $A-(-4)I = A+4I$

$$=\begin{bmatrix}3 & -1 & 3\\ -1 & 3 & 3\\ 6 & 6 & 2\end{bmatrix}+\begin{bmatrix}4 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 4\end{bmatrix}=\begin{bmatrix}7 & -1 & 3\\ -1 & 7 & 3\\ 6 & 6 & 6\end{bmatrix}$$

The augmented matrix for $(A-(-4)I)\vec{x}=\vec{0}$ is

$$\begin{bmatrix}7 & -1 & 3 & 0\\ -1 & 7 & 3 & 0\\ 6 & 6 & 6 & 0\end{bmatrix}\sim\begin{bmatrix}1 & 1 & 1 & 0\\ -1 & 7 & 3 & 0\\ 7 & -1 & 3 & 0\end{bmatrix}\sim\begin{bmatrix}1 & 1 & 1 & 0\\ 0 & 8 & 4 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 1 & 1 & 0\\ 0 & 1 & \frac{1}{2} & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\sim\begin{bmatrix}1 & 0 & \frac{1}{2} & 0\\ 0 & 1 & \frac{1}{2} & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

So the solution is $\vec{x}=x_3\begin{bmatrix}-\frac{1}{2}\\ -\frac{1}{2}\\ 1\end{bmatrix}$

A basis for the eigenspace corresponding to the eigenvalue $\lambda=-4$ is $\left\{\begin{bmatrix}-\frac{1}{2}\\ -\frac{1}{2}\\ 1\end{bmatrix}\right\}$

Theorem 1

The eigenvalues of a triangular matrix are the entries on its main diagonal.

Eg: $A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix}$. Then $A-2I$, $A-4I$, $A-5I$ all have less than 3 pivot positions.

This means the equations $(A-2I)\vec{x} = \vec{0}$, $\cdots$ have nontrivial solutions.

Example 3. Find the eigenvalues of the given matrix.

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -4 \end{bmatrix}$$

By Thm 1, the eigenvalues are $2, 0, -4$.

Example 4. For $A = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 3 & 5 \\ 1 & 3 & 5 \end{bmatrix}$, find one eigenvalue, with no calculation. Justify your answer.

Discussion: When does A has an eigenvalue 0?

A has an eigenvalue 0

$\Longleftrightarrow$ $A\vec{x} = 0\vec{x} = \vec{0}$ has a nontrivial solution (by def).

$\Longleftrightarrow$ A is not invertible (by the invertible matrix theorem)

Thus A has an eigenvalue 0 $\Longleftrightarrow$ A is not invertible.

ANS: Notice that the columns of the given A are linearly dependent. So A is not invertible.

Thus 0 must be an eigenvalue for A.

Theorem 2

If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent.

Exercise 5. Without calculation, find one eigenvalue and two linearly independent eigenvectors of $A = \begin{bmatrix} 3 & 3 & -3 \\ 3 & 3 & -3 \\ 3 & 3 & -3 \end{bmatrix}$. Justify your answer.

Solution. The matrix $A = \begin{bmatrix} 3 & 3 & -3 \\ 3 & 3 & -3 \\ 3 & 3 & -3 \end{bmatrix}$ is not invertible because its columns are linearly dependent.

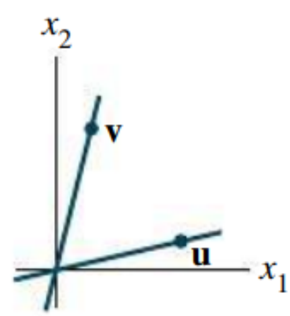
So the number 0 is an eigenvalue of A (see the discussion in **Example 4**).

Eigenvectors for the eigenvalue 0 are solutions of $A\mathbf{x} = \mathbf{0}$ and therefore have entries that produce a linear dependence relation among the columns of A.

Any nonzero vector (in $\mathbb{R}^3$) whose first and second entries, minus the third, sum to 0, will work.

Find any two such vectors that are not multiples; for instance, $(1, 0, 1)$ and $(0, 1, 1)$.

Exercise 6. Let $\mathbf{u}$ and $\mathbf{v}$ be the vectors shown in the figure, and suppose $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors of a 2×2 matrix A that correspond to eigenvalues -2 and 4, respectively. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$ for each $\mathbf{x}$ in $\mathbb{R}^2$, and let $\mathbf{w} = \mathbf{u} + \mathbf{v}$. Make a copy of the figure, and on the same coordinate system, carefully plot the vectors $T(\mathbf{u}), T(\mathbf{v})$, and $T(\mathbf{w})$.



ANS: We know $\vec{u}, \vec{v}$ are eigenvectors corresponding to eigenvalues -2, 4, respectively

$T(\vec{u}) = A\vec{u} = -2\vec{u}$

$T(\vec{v}) = A\vec{v} = 4\vec{v}$

As $\vec{w} = \vec{u} + \vec{v}$, $T(\vec{w}) = A\vec{u} + A\vec{v} = -2\vec{u} + 4\vec{v}$.

We plot the vectors $T(\vec{u}) = -2\vec{u}$, $T(\vec{v}) = 4\vec{v}$, $T(\vec{w}) = -2\vec{u} + 4\vec{v}$